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Completeness of a set of modes connected with the electromagnetic field of a homogeneous sphere embedded in an infinite medium

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Abstract. In this paper we consider the mode decomposition of the electromagnetic field connected with a sphere of conductivity σ_1, and dielectric and magnetic permeabilities ϵ_1 and μ_1 respectively. The sphere is embedded in an infinite medium, characterised by the constants σ_2, ϵ_2, and μ_2. The continuity requirements of the field across the surface of the sphere lead to a certain set of so called natural modes, which can be calculated explicitly if the radial part of the electric field strength inside the sphere equals zero. The completeness of the radial parts of these modes, which is a set of spherical Bessel functions, is sometimes erroneously deduced from Sturm–Liouville theory. This theory however cannot be used to show the completeness because the *continuity* conditions of the field lead to a boundary value problem with a boundary condition which explicitly depends on the eigenvalue. The completeness of this set of functions, which is necessary to solve an initial value problem, will be shown. The set of functions will even be shown to be overcomplete. The connection of this problem with many other similar problems occurring in mathematical physics, as well as the physical consequences of the overcompleteness, will be discussed.

1. Introduction

Many boundary value problems in theoretical physics are special cases of Sturm–Liouville (SL) theory from which the possibility of expanding an 'arbitrary' function into a set of modes satisfying a linear differential equation with homogeneous linear boundary conditions is deduced (Courant and Hilbert 1966). However ordinary SL theory only applies to *boundary value* problems, requiring that a linear combination of the field and its normal derivative at a boundary be equal to zero, whereas in physics one often has to deal with *continuity* problems. These problems are connected with the continuity conditions of a field across a boundary and lead usually to a 'boundary' condition which explicitly contains the eigenvalue (equation (2.11) of this paper). The explicit occurrence of an eigenvalue in the boundary condition(s) is not incorporated in classical SL theory and profoundly affects the results of this theory. For instance, a set of natural modes satisfying a linear differential equation and such a 'boundary' condition can be overcomplete, whereas the eigenvalues usually have a non-vanishing imaginary part. Both phenomena do not occur in SL theory (Courant and Hilbert 1966). The imaginary part of the eigenvalues leads to damped vibrations, so that the total amount of radiated energy is finite (Bremmer 1949, and Bateman 1955).

An example of such a continuity problem is the calculation of the electromagnetic field inside and outside a sphere with radius a and conductivity σ_1, dielectric and magnetic permeabilities ϵ_1 and μ_1, embedded in an infinite medium characterised by σ_2, ϵ_2, and μ_2, where $\sigma_1, \epsilon_1, \ldots, \mu_2$ are constants. The quantities which according to electromagnetic theory are continuous across the boundary between the sphere and the medium are the tangential components of the electric and magnetic field vectors $\boldsymbol{n} \times \boldsymbol{E}$ and $\boldsymbol{n} \times \boldsymbol{H}$. This leads to a set of natural modes and a transcendental equation for the allowed values of the frequency. *Assuming* the completeness of the set of natural modes, and in particular a set of spherical Bessel functions, Stratton (1941) solves the following initial value problem. Determine the electromagnetic field inside the sphere for all values of $t > 0$ if the radial part of the magnetic field vector inside the sphere is given at $t = 0$ and the radial part of the electric field vector vanishes identically. (See § 2 of this paper for explicit calculations.)

Stratton states that the completeness of this set of spherical Bessel functions follows from the theory of Fourier–Bessel series (Watson 1966) which is just a special case of SL theory. But, this statement is, unfortunately, not valid because the 'boundary' condition which is derived from the continuity condition, explicitly depends upon the eigenvalue. (The set of modes is even shown to be linearly dependent!) In this paper we will obtain a proof for the completeness of this set of functions, together with an expansion formula.

The completeness problem considered in this paper, which in the first instance looks rather special, arises in many different branches of physics if one solves initial value or scattering problems. Moreover, several sets of natural modes and the completeness problems generated by them have been known for a long time. We will conclude this introduction with several examples of sets of natural modes and show their usefullness for solving initial value or scattering problems. The related completeness problems will be formulated as well.

For instance, the set of modes considered in this paper also arises in connection with the theory of scattering of a plane wave by a sphere (Mie scattering), i.e. the numerators of the amplitudes of this type of scattered field are equivalent to the 'boundary' condition of this paper (Born and Wolf 1975, § 13.5, see especially Bromwich 1919), and a partial fraction Cauchy-type expansion of this field inside the sphere if the conductivities are zero leads to an infinite series of natural modes. (The details of this calculation can be found in the book by Nussenzveig (1972, § 5.7c) who derived a partial fraction expansion of the S matrix.) The calculation of a scattered field arising from the scattering of an arbitrary incoming wave by a sphere as a series expansion of the natural modes of a sphere was indicated by Bateman (1955). Bremmer (1949) suggested that a scattering theory in terms of natural modes would lead to a promising alternative with respect to ordinary scattering theory.

Moreover, as early as 1884, Thomson (1884), and somewhat later Love (1904) used the natural modes of the electromagnetic field outside a perfectly conducting sphere to determine the field outside this sphere if at $t = 0$ a charge distribution is given at the surface of the sphere (see especially Beck and Nussenzveig 1960).

A well known example of a set of natural modes is generated by the theory of potential scattering (Siegert 1939, Humblet and Rosenfeld 1961, Pattanayak 1976). This set arises in connection with the calculation of the field scattered by a central potential with finite support. The 'boundary' condition of this problem is in fact derived from the continuity requirements of the field and its normal derivative across the boundary between the potential and free space. The eigenfrequencies k_n are complex and lead to damped vibrations (Humblet and Rosenfeld 1961).

The natural modes of this problem are the solutions of a linear homogeneous integral equation arising in the theory of potential scattering:

$$\phi_n(\boldsymbol{r})=\int_\tau G(\boldsymbol{r}, \boldsymbol{r}'; k_n)U(\boldsymbol{r}')\phi_n(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}', \tag{1.1}$$

if

$$G(\boldsymbol{r}, \boldsymbol{r}'; k)=\frac{\exp(\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|)}{|\boldsymbol{r}-\boldsymbol{r}'|}, \qquad U(\boldsymbol{r}')=\frac{2m}{\hbar^2}V(\boldsymbol{r}') \tag{1.2}$$

and $V(\boldsymbol{r})$ denotes the potential and τ the support of V. Equation (1.1) shows that the natural modes $\phi_n(\boldsymbol{r})$ are solutions of the time-independent Schrödinger equation, subject to the *non-local* boundary condition

$$\int_\sigma\left(\phi_n(\boldsymbol{r})\frac{\partial}{\partial n}G(\boldsymbol{r}, \boldsymbol{r}'; k_n)-\frac{\partial}{\partial n}\phi_n(\boldsymbol{r})G(\boldsymbol{r}, \boldsymbol{r}'; k_n)\right)\mathrm{d}\sigma=0, \tag{1.3}$$

which has to be satisfied everywhere inside the boundary σ of the support of V (Pattanayak 1976).

The scattering problem can be solved if it can be shown that the natural modes satisfy the completeness relation

$$\sum_n \phi_n(\boldsymbol{r})\psi_n(\boldsymbol{r}')=\delta(\boldsymbol{r}-\boldsymbol{r}')U(\boldsymbol{r}'). \tag{1.4}$$

The relation (1.4) has been established by Hoenders (1977), and the solution of the integral equation of potential scattering reads as

$$\psi(\boldsymbol{r})=\psi^{\mathrm{inc}}(\boldsymbol{r})+\sum_n\int_\tau\frac{\phi_n(\boldsymbol{r})\psi_n(\boldsymbol{r}')}{k^2-k_n^2}\psi^{\mathrm{inc}}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}', \qquad \boldsymbol{r}\in\tau, \boldsymbol{r}\notin\sigma. \tag{1.5}$$

(It is readily observed from (1.4) that (1.5) is a solution of the time-independent Schrödinger equation.) The wavefunction outside τ is obtained taking the limit $\boldsymbol{r}\to\sigma$ in equation (1.5) and solving the exterior boundary value problem of the Helmholtz equation, subject to Sommerfeld's radiation condition. This type of analysis has been conjectured by More and Gerjuoy (1973). A similar theory has been developed for the theory of electromagnetic scattering (Hoenders 1977).

A general definition of the natural modes of an electromagnetic or scalar field arising in connection with scattering problems has been given by Wolf and Pattanayak (1976, see also Agarwal 1973 for many applications). They defined the natural modes for a body of any prescribed constitutive relations bounded by a closed surface S as the set of well behaved outgoing solutions of Maxwell's equations, which obey the continuity conditions at the boundary S. (A similar definition can be formulated for the natural modes of potential scattering.) Wolf and Pattanayak (1976) showed that the solutions of Maxwell's equation satisfying this definition are the eigenfunctions of the homogeneous part of the integro-differential equations of electromagnetic scattering with eigenvalues $k=k_n$, or, solutions to Maxwell's equations satisfying a non-local boundary condition. These two formulations are equivalent.

The connection between this definition and the sets of natural modes generated by Mie scattering has been established by Agarwal, whereas Pattanayak (1976) showed that this general definition leads to the natural modes of potential scattering.

We end this short survey on the occurrence of sets of natural modes in physics with an example due to Morse and Feshbach (1953). They considered an initial value problem connected with a string with one non-rigid support. This support leads to a *continuity condition* similar to equation (2.2*b*), rather than to a boundary condition, and the solution of this problem is obtained in terms of a series expansion of natural modes. (Their solution unfortunately contains an error because the calculation of the residues of the integral occurring above equation (11.1.27) is wrong. The correct solution of the problem can be obtained by either the methods of this paper or from an expansion due to Geppert (1924, § 3, example 2). Similar mechanical problems are considered by Lamb (1900) (see Nussenzveig 1972, § 4.4, and Beck and Nussenzveig 1960).

The examples of Love (1904), Thomson (1884), Lamb (1900), Nussenzveig (1972), Pattanayak (1976), Humblet and Rosenfeld (1961), Morse and Feshbach (1953), Beck and Nussenzveig (1960), Bateman (1955) and Agarwal (1973), show that sets of natural modes occur everywhere in physics and are the rule rather than the exception. A systematic analysis of their properties is therefore very desirable. General results have been obtained by Hoenders (1977) who developed a Hilbert–Schmidt type of theory, leading to a bilinear expansion in terms of the natural modes of the resolvent kernel connected with the integral equations of electromagnetic and potential scattering theory (see equation (1.2)).

The initial value problem considered by Stratton could be solved using this Hilbert–Schmidt type of theory taking the temporal Laplace transform of the Maxwell equations. Though this procedure would lead to an indirect proof of the completeness of the set of natural modes we prefer to give an entirely different and direct proof of the desired completeness. The reason for doing this is that the techniques used in this proof are very general and can be applied to other expansion problems with boundary conditions which are different from the ordinary Sturm–Liouville boundary conditions or the continuity conditions to be imposed on the solutions of the Maxwell equations or the Schrödinger equation. Examples of such problems are the problems considered by Morse and Feshbach (1953) and Lamb (1900), mentioned above.

The completeness proof is obtained using a very elegant and beautiful method employing the calculus of residues used for the first time by Cauchy (1827 a, b, c), and developed by numerous other authors (e.g. Poincaré 1894, Birkhoff 1908, Hilb 1918, Tamarkine 1927, Geppert 1924, 1925, Titchmarsh 1970).

This technique, applied to SL-type problems with boundary conditions depending upon the eigenvalue, may provide a tool with which a large class of similar problems can be handled.

2. Statement of the problem

Suppose a sphere of radius a and characterised electrically by the constant $k_1^2 = \epsilon_1\mu_1\omega^2 + i\sigma_1\mu_1\omega$ is embedded in an infinite homogeneous medium characterised by the parameter $k_2^2 = \epsilon_2\mu_2\omega^2 + i\sigma_2\mu_2\omega$, where ϵ_1, μ_1 and σ_1 are the dielectric permeability, the magnetic permeability, and conductivity of the sphere respectively, $\epsilon_2, \mu_2, \sigma_2$, the corresponding quantities for the infinite surrounding medium, and ω is the temporal frequency of a Fourier component of the field. At time $t = 0$ let the radial component $H_r^{(i)}(R, \theta, \phi)$ of the magnetic field inside the sphere be given by some arbitrary function $f(R, \theta, \phi)$ which, for mathematical convenience, is supposed to be of bounded variation. Then, if we consider those modes for which the radial component of the electric field inside the medium equals zero, i.e. we only consider transverse electric waves, the field, both inside and outside the sphere, is determined uniquely by $f(R, \theta, \phi)$, and can be calculated using the natural modes of the sphere (Stratton 1941, § 9.22). Following Stratton's treatment we have the following set of modes inside the medium:

(a) $$E_r^{(i)} = 0,$$

(b) $$E_\theta^{(i)} = -\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} Y_{mn} j_n(k_1 R)\,\mathrm{e}^{-\mathrm{i}\omega t},$$

(c) $$E_\phi^{(i)} = \frac{\partial}{\partial\theta} Y_{mn} j_n(k_1 R)\,\mathrm{e}^{-\mathrm{i}\omega t},$$

(d) $$H_r^{(i)} = -\frac{n(n+1)}{\mathrm{i}\omega\mu_1} Y_{mn}\frac{j_n(k_1 R)}{R}\,\mathrm{e}^{-\mathrm{i}\omega t},$$

(e) $$H_\theta^{(i)} = -\frac{1}{\mathrm{i}\omega\mu_1}\frac{\partial Y_{mn}}{\partial\theta}\frac{1}{R}(k_1 R j_n(k_1 R))'\,\mathrm{e}^{-\mathrm{i}\omega t},$$

(f) $$H_\phi^{(i)} = -\frac{1}{\mathrm{i}\omega\mu_1}\frac{1}{\sin\theta}\frac{\partial Y_{mn}}{\partial\phi}\frac{1}{R}(k_1 R j_n(k_1 R))'\,\mathrm{e}^{-\mathrm{i}\omega t}. \tag{2.1}$$

The prime denotes differentiation with respect to $(k_1 R)$, j_n is the spherical Bessel function of order n, and Y_{mn} the spherical harmonic. The field in the external region $R > a$ is obtained by replacing k_1 by k_2, and j_n by $h_n^{(1)}$, the spherical Hankel function of order n of the first kind. In order to satisfy the boundary conditions that $\boldsymbol{n}\times\boldsymbol{H}$ and $\boldsymbol{n}\times\boldsymbol{E}$ are continuous, the frequency ω can only take a discrete set of values ω_{sn} which have to be determined from the equation

$$\rho_{sn}^2 = (\omega_{sn}^2\epsilon_2\mu_2 + \mathrm{i}\omega_{sn}\mu_2\sigma_2)a^2, \tag{2.2a}$$

where the numbers ρ_{sn} are the roots of the following transcendental equation:

$$\mu_2(N\rho j_n(N\rho))' h_n^{(1)}(\rho) - \mu_1(\rho h_n^{(1)}(\rho))' j_n(N\rho) = 0, \tag{2.2b}$$

$$\rho = k_2 a; \qquad k_1 = N k_2; \qquad k_1 a = N\rho, \tag{2.3}$$

where the prime denotes differentiation with respect to the argument.

Assuming that the set of functions

$$\{j_n(k_{sn}R)\}; \qquad 0 < R < a,\ k_{sn} = \frac{N}{a}\rho_{sn}, \tag{2.4}$$

is complete, Stratton shows how the field can be constructed from the modes.

The solution of his initial value problem is obtained from the expansion of the value $f(R, \theta, \phi)$ at $t = 0$ of $H_r^{(i)}$ into a series of spherical harmonics:

$$f(R, \theta, \phi) = \sum_{m=0}^{\infty}\sum_{n=-m}^{+m} a_{nm}(R) Y_{mn}(\theta, \phi), \tag{2.5}$$

if

$$a_{nm}(R)=\int_{\Omega}\mathrm{d}\Omega\, f(R,\theta,\phi)Y_{mn}(\theta,\phi). \tag{2.6}$$

On expanding the functions $Ra_{nm}(R)$ into the set of functions

$$Ra_{nm}(R)=\sum_{s}c_{nm}(s)j_n(k_{sn}R), \tag{2.7}$$

we obtain from equations (2.1d), (2.5), (2.6) and (2.7):

$$H_r^{(\mathrm{i})}(R,\theta,\phi,t)$$
$$=\sum_{m=0}^{\infty}\sum_{n=-m}^{+m}\sum_{s}-\frac{n(n+1)}{\mathrm{i}\omega_{sn}\mu_1}Y_{mn}(\theta,\phi)c_{nm}^{(1)}(s)\frac{j_n(k_{sn}R)}{R}\mathrm{e}^{-\mathrm{i}\omega_{sn}t}, \tag{2.8}$$

if

$$c_{nm}^{(1)}(s)=\frac{\mathrm{i}\omega_{sn}\mu_1}{n(n+1)}. \tag{2.9}$$

However, from (2.2b) we observe that we cannot prove the completeness of the set of functions (2.4) by showing that they satisfy an ordinary Sturm–Liouville expansion problem as stated by Stratton, because the Sturm–Liouville approach would be to show the completeness of the set of functions which are regular near the origin and satisfy the differential equation

$$\left(R^2\frac{\mathrm{d}^2}{\mathrm{d}R^2}+2R\frac{\mathrm{d}}{\mathrm{d}R}+(\rho^2N^2a^{-2}R^2-n^2)\right)j_n(\rho Na^{-1}R)=0, \tag{2.10}$$

and the boundary condition

$$\mu_2A(\rho)\frac{\mathrm{d}}{\mathrm{d}(N\rho)}(N\rho j_n(N\rho))-\mu_1B(\rho)j_n(N\rho)=0, \tag{2.11}$$

where

$$A(\rho)=h_n^{(1)}(\rho), \tag{2.12}$$

$$B(\rho)=\frac{\mathrm{d}}{\mathrm{d}\rho}(\rho h_n^{(1)}(\rho)). \tag{2.13}$$

However, in contrast to ordinary Sturm–Liouville theory the coefficients of the boundary condition (2.11) depend explicitly upon the eigenvalue ρ, which circumstance completely changes the applicability of the results of ordinary Sturm–Liouville theory. In particular the property of orthogonality of eigenfunctions will no longer be valid (see Morse and Feshbach 1953). As already remarked, this kind of boundary condition is typical for electromagnetic theory, where the 'boundary' conditions are continuity conditions.

3. Calculational procedure

For simplicity we will first consider the case of zero conductivities σ_1 and σ_2. The proof for the completeness of the modes if σ_1 and σ_2 are not zero will be postponed till

§ 4. The reason for doing this is that for zero conductivities the quantity N equals $(\epsilon_1\mu_1)^{1/2}(\epsilon_2\mu_2)^{-1/2}$ and is therefore a constant independent of ρ. Let

$$\psi(\rho, n)=\rho[\mu_2(N\rho j_n(N\rho))'h_n^{(1)}(\rho)-\mu_1(h_n^{(1)}(\rho)\rho)'j_n(N\rho)]. \tag{3.1}$$

We will assume that the roots of the equation $\psi(\rho, n)=0$ are simple. A proof of this conjecture is rather difficult and is connected with the multiplicity of the poles of the S matrix (Nussenzveig 1972, § 5.7a).

The spherical Bessel and Hankel functions can be written as (Stratton 1941, § 7.4, Magnus and Oberhettinger 1966):

$$\begin{aligned}j_n(\rho)&=\rho^{-1}\Bigg(\sin(\rho-\tfrac{1}{2}n\pi)\sum_{m=0}^{\leq\frac{1}{2}n}(-1)^m(n+\tfrac{1}{2},2m)(2\rho)^{-2m}+\cos(\rho-\tfrac{1}{2}n\pi)\\&\quad\times\sum_{m=0}^{\leq\frac{1}{2}n-\frac{1}{2}}(-1)^m(n+\tfrac{1}{2},2m+1)(2\rho)^{-2m-1}\Bigg)\\&=(-1)^n\rho^n\left(\frac{\mathrm{d}}{\rho\,\mathrm{d}\rho}\right)^n\frac{\sin\rho}{\rho},\end{aligned} \tag{3.2a}$$

$$\begin{aligned}y_n(\rho)&=\rho^{-1}\Bigg(-\cos(\rho-\tfrac{1}{2}n\pi)\sum_{m=0}^{\leq\frac{1}{2}n}(-1)^m(n+\tfrac{1}{2},2m)(2\rho)^{-2m}+\sin(\rho-\tfrac{1}{2}n\pi)\\&\quad\times\sum_{m=0}^{\leq\frac{1}{2}n-\frac{1}{2}}(-1)^m(n+\tfrac{1}{2},2m+1)(2\rho)^{-2m-1}\Bigg)\\&=-(-1)^n\rho^n\left(\frac{\mathrm{d}}{\rho\,\mathrm{d}\rho}\right)^n\frac{\cos\rho}{\rho},\end{aligned} \tag{3.2b}$$

$$h_n^{(1),(2)}(\rho)=j_n(\rho)\pm\mathrm{i}y_n(\rho)=\mp\mathrm{i}(-1)^n\rho^n\left(\frac{d}{\rho\,\mathrm{d}\rho}\right)^n\frac{\mathrm{e}^{\pm\mathrm{i}\rho}}{\rho}, \tag{3.2c}$$

where

$$\begin{aligned}&(\lambda, n)=2^{-2n}n!\prod_{k=1}^{n}[4\lambda^2-(2k-1)^2] \qquad \text{(Hankel's symbol)},\\&(\lambda, 0)=1.\end{aligned} \tag{3.2d}$$

Equations (3.2*a*) to (3.2*c*) show the existence of complex numbers a_j and b_j such that:

$$\psi(\rho, n)=\exp(\mathrm{i}\rho)\left(\cos(\rho N)\sum_{j=0}^{2n+2}a_j\rho^{-j}+\sin(\rho N)\sum_{j=0}^{2m+2}b_j\rho^{-j}\right). \tag{3.3}$$

Let

$$\phi(\rho, n)=\exp(\mathrm{i}\rho)\left(-\sin(\rho N)\sum_{j=0}^{2n+2}a_j\rho^{-j}+\cos(\rho N)\sum_{j=0}^{2n+2}b_j\rho^{-j}\right), \tag{3.4}$$

and suppose that the function $f(\tau)$ is defined on the interval $[0, a]$, satisfying Dirichlet's conditions: e.g. $f(\tau)$ is of bounded variation on $[0, a]$. Let

$$f_1(\rho, R)=\rho^2\phi(\rho, n)j_n(\rho RNa^{-1})\int_0^R\tau^2j_n(\rho\tau Na^{-1})f(\tau)\,\mathrm{d}\tau, \tag{3.5a}$$

$$f_2(\rho, R)=\rho^2\phi(\rho, n)j_n(\rho RNa^{-1})\int_R^a\tau^2j_n(\rho\tau Na^{-1})f(\tau)\,\mathrm{d}\tau, \tag{3.5b}$$

$$f_3(\rho, R)=\rho^2 y_n(\rho R N a^{-1}) \int_0^R \tau^2 j_n(\rho N a^{-1}\tau) f(\tau)\,\mathrm{d}\tau, \tag{3.5c}$$

and

$$f_4(\rho, R)=\rho^2 j_n(\rho R N a^{-1}) \int_R^a \tau^2 y_n(\rho N a^{-1}\tau) f(\tau)\,\mathrm{d}\tau. \tag{3.5d}$$

Consider the following two contour integrals for large values of the positive number c:

$$I_1(R, c)=\frac{1}{2\pi \mathrm{i}} \oint_{|\rho|=c} \left(\frac{f_1(\rho, R)}{\psi(\rho, n)}-f_3(\rho, R)\right) \mathrm{d}\rho, \tag{3.6}$$

$$I_2(R, c)=\frac{1}{2\pi \mathrm{i}} \oint_{|\rho|=c} \left(\frac{f_2(\rho, R)}{\psi(\rho, n)}-f_4(\rho, R)\right) \mathrm{d}\rho, \tag{3.7}$$

which will be evaluated with the theorem of residues and also by using the asymptotic behaviour of the integrand on the contour. The contour is chosen such that it passes between two successive zeros of the denominator.

The second terms in the integrands of equations (3.6) and (3.7) are entire functions of ρ and therefore give no contribution to the contour integrals. However, the insertion of these functions makes the asymptotic evaluation of the contour integrals feasible.

From the theorem of residues we obtain (see the remark following (3.1)):

$$I_1(R, c)+I_2(R, c)=\sum_s \frac{\phi(\rho_{sn}, n)\rho_{sn}^2}{(\psi(\rho_{sn}, n))'} j_n(k_{sn}R) \int_0^a \tau^2 j_n(k_{sn}\tau) f(\tau)\,\mathrm{d}\tau) \tag{3.8}$$

where the summation has to be extended over all the singularities of the integrands of equations (3.6) and (3.7) lying inside the contour. The point $\rho=0$ gives no contribution to the integrals (3.6) and (3.7) because though $\phi(\rho, n)=\mathrm{O}(\rho^{-2n-2})$ the function $\rho^2 j_n(\rho N a^{-1} R) j_n(\rho N a^{-1})=\mathrm{O}(\rho^{2n+2})$ near $\rho=0$. We will also evaluate the left-hand side of (3.8) in the limit $c\to\infty$ by using the asymptotic behaviour of the integrands. To this end we have to study the asymptotics of the factors of the integrands. Firstly we study $\phi(\rho, n)\psi(\rho, n)$.

From equations (3.3) and (3.4) we obtain for large values of c:

$$\frac{\phi(\rho, n)}{\psi(\rho, n)}=\begin{cases} +\mathrm{i}+\mathrm{O}(\exp(2\mathrm{i}\rho N)), & \text{if } 0\leqslant \arg\rho\leqslant\pi \quad (3.9a)\\ -\mathrm{i}+\mathrm{O}(\exp(-2\mathrm{i}\rho N), & \text{if } \pi\leqslant \arg\rho\leqslant 2\pi \quad (3.9b)\end{cases}$$

Combination of equations (3.2c), (3.5a) to (3.5d), (3.9a) and (3.9b) leads to:

$$\begin{aligned}\frac{f_1(\rho, R)}{\psi(\rho, n)}&-f_3(\rho, R)\\ &=\mathrm{i}\rho^2 h_n^{(1)}(\rho N a^{-1} R) \int_0^R \tau^2 j_n(\rho N a^{-1}\tau) f(\tau)\,\mathrm{d}\tau\\ &\quad\times\left[1+\mathrm{O}\left(\rho^2 j_n(\rho R N a^{-1}) \int_0^R \tau^2 j_n(\rho N a^{-1}\tau) f(\tau)\,\mathrm{d}\tau \exp(2\mathrm{i}\rho N)\right)\right],\\ &\quad \text{if } 0\leqslant\arg\rho\leqslant\pi \end{aligned} \tag{3.10a}$$

$$= -\mathrm{i}\rho^2 h_n^{(2)}(\rho Na^{-1}R)\int_0^R \tau^2 j_n(Na^{-1}\tau)f(\tau)\,\mathrm{d}\tau$$
$$\times\left[1+\mathrm{O}\left(\rho^2 j_n(\rho RNa^{-1})\int_0^R \tau^2 j_n(\rho Na^{-1}\tau)f(\tau)\,\mathrm{d}\tau\,\exp(-2\mathrm{i}\rho N)\right)\right],$$
$$\text{if } \pi \leqslant \arg\rho \leqslant 2\pi \tag{3.10b}$$

$$\frac{f_2(\rho,R)}{\psi(\rho,n)} - f_4(\rho,R)$$
$$= \mathrm{i}\rho^2 j_n(\rho Na^{-1}R)\int_R^a \tau^2 h_n^{(1)}(\rho Na^{-1}\tau)f(\tau)\,\mathrm{d}\tau$$
$$\times\left[1+\mathrm{O}\left(\rho^2 j_n(\rho RN_a^{-1})\int_R^a \tau^2 j_n(\rho Na^{-1}\tau)f(\tau)\mathrm{d}\tau\,\exp(2\mathrm{i}\rho N)\right)\right],$$
$$\text{if } \pi \leqslant \arg\rho \leqslant \pi \tag{3.10c}$$
$$= -\mathrm{i}\rho^2 j_n(\rho Na^{-1}R)\int_R^a \tau^2 h_n^{(2)}(\rho Na^{-1}\tau)f(\tau)\,\mathrm{d}\tau$$
$$\times\left[1+\mathrm{O}\left(\rho^2 j_n(\rho RNa^{-1})\int_R^a \tau^2 j_n(\rho Na^{-1}\tau)f(\tau)\mathrm{d}\tau\,\exp(-2\mathrm{i}\rho N)\right)\right],$$
$$\text{if } \pi \leqslant \arg\pi \leqslant 2\pi. \tag{3.10d}$$

In order to carry the asymptotics further we have to study the integrals

$$\int_0^R j_n(\rho Na^{-1}\tau)f(\tau)\,\mathrm{d}\tau \qquad\text{and}\qquad \int_R^a h_n^{(1)}(\rho Na^{-1}\tau)f(\tau)\,\mathrm{d}\tau.$$

We will first derive a useful identity. From the recurrence relation for spherical Bessel functions (Stratton 1941, § 7.4)

$$\frac{\mathrm{d}}{\mathrm{d}\rho}(\rho^{n+1}z_n(\rho)) = \rho^{n+1}z_{n-1}(\rho), \tag{3.11}$$

where $z_n(\rho)$ denotes any spherical Bessel function of order n we obtain

$$\int_a^b \tau^2 z_n(\rho Na^{-1}\tau)\,\mathrm{d}\tau$$
$$= \frac{1}{\rho Na^{-1}}\int_a^b \tau^{-n}\frac{\mathrm{d}}{\mathrm{d}\tau}(\tau^{n+2}z_{n+1}(\rho Na^{-1}\tau))\,\mathrm{d}\tau$$
$$= \frac{1}{\rho Na^{-1}}\left(\tau^2 z_{n+1}(\rho Na^{-1}\tau)\Big|_a^b + n\int_a^b \tau z_{n+1}(\rho Na^{-1}\tau)\,\mathrm{d}\tau\right), \tag{3.12}$$

if a and b are real numbers, and provided that all the integrals exist. Equation (3.12) is valid if $|\rho| \neq 0$, $0 \leqslant \arg\rho \leqslant 2\pi$ and will be very useful for the calculation of the asymptotic behaviour of the integrals mentioned previously. Because the function $f(\tau)$ is of bounded variation and therefore only discontinuous at a finite number of points in the interval $0 \leqslant \tau \leqslant a$ there exist to every positive number ϵ intervals $\delta(\epsilon)$ such that:

$$|f(R)-f(\tau)| < \epsilon, \qquad \text{if } 0 < R-\delta \leqslant \tau \leqslant R, \qquad R \leqslant \tau \leqslant R+\delta < a, \tag{3.13}$$

i.e. the property (3.13) allows $f(\tau)$ to be discontinuous at $\tau = R$. Suppose that we only consider those values of $|\rho|=c$ for which $c>(R-\delta)^{-1}$, then:

$$\int_0^R \tau^2 j_n(\rho Na^{-1}\tau)f(\tau)\,d\tau=\int_0^{c^{-1}}+\int_{c^{-1}}^{R-\delta}+\int_{R-\delta}^{R}\tau^2 j_n(\rho Na^{-1}\tau)f(\tau)\,d\tau. \tag{3.14}$$

The first term of the power series expansion of the entire function $j_n(x)$ is proportional to x^n, therefore

$$\begin{aligned}\left|\int_0^{c^{-1}}\tau^2 j_n(\rho Na^{-1}\tau)f(\tau)\,d\tau\right| &= O\left(\int_0^{c^{-1}}\tau^2(\rho Na^{-1}\tau)^n f(\tau)\,d\tau\right)\\ &= O\left(\rho^{-3}\int_0^1 y^{n+2}\,dy\right)=O(\rho^{-3}), \qquad \text{if } 0\leqslant\arg\rho\leqslant 2\pi,\ |\rho|=c.\end{aligned} \tag{3.15}$$

Using equations (3.2a) we obtain:

$$\left|\int_{c^{-1}}^{R-\delta}\tau^2 j_n(\rho Na^{-1}\tau)f(\tau)\,d\tau\right|=O\{\exp[|cNa^{-1}(R-\delta)\sin(\arg\rho)|]\}. \tag{3.16}$$

Consider the identity

$$\begin{aligned}\int_{R-\delta}^{R}\tau^2 j_n(\rho Na^{-1}\tau)f(\tau)\,d\tau &= f(R)\int_{R-\delta}^{R}\tau^2 j_n(\rho Na^{-1}\tau)\,d\tau+\int_{R-\delta}^{R}\tau^2 j_n(\rho Na^{-1}\tau)(f(\tau)-f(R))\,d\tau.\end{aligned} \tag{3.17}$$

From equation (3.12) we derive

$$\left|f(R)\int_{R-\delta}^{R}\tau^2 j_n(\rho Na^{-1}\tau)\,d\tau-f(R)\frac{R^2a}{\rho N}j_{(n+1)}(\rho Na^{-1}R)\right|=O[\rho^{-3}\exp(|\rho Na^{-1}R|)]. \tag{3.18}$$

Let

$$P(\tau,\rho)=\mathrm{Re}(j_n(\rho Na^{-1}\tau)), \tag{3.19a}$$

and

$$Q(\tau,\rho)=\mathrm{Im}(j_n(\rho Na^{-1}\tau)). \tag{3.19b}$$

According to the second mean value theorem there exists numbers ξ' and ξ'' such that:

$$\begin{aligned}\int_{R-\delta}^{R}(f(\tau)-f(R))\tau^2 j_n(\rho Na^{-1}\tau)\,d\tau &= (f(R-\delta)-f(R))\left(\int_{R-\delta}^{\xi'}P(\tau,\rho)\,d\tau+i\int_{R-\delta}^{\xi''}Q(\tau,\rho)\,d\tau\right),\\ &\text{if } R-\delta<\xi'<R, \qquad R-\delta<\xi''<R.\end{aligned} \tag{3.20}$$

Combination of (3.12), (3.13) and (3.20) leads to

$$\left|\int_{R-\delta}^{R}(f(\tau)-f(R))\tau^2 j_n(\rho Na^{-1}\tau)\,d\tau\right|=O[\epsilon\rho^{-2}\exp(\rho Na^{-1}R)], \tag{3.21}$$

and combination of (3.15), (3.16), (3.17), (3.18) and (3.21) leads to

$$\left|\int_0^R \tau^2 j_n(\rho N a^{-1}\tau) f(\tau)\,\mathrm{d}\tau - f(R)\frac{R^2 a}{\rho N} j_{n+1}(\rho N a^{-1} R)\right|$$
$$= \mathrm{O}[\epsilon\rho^{-2}\exp(Na^{-1}R)], \qquad \text{if } 0 \leqslant \arg\rho \leqslant 2\pi \quad \text{and} \quad |\rho| \to \infty. \tag{3.22}$$

Similarly we obtain:

$$\left|\int_R^a \tau^2 h_n^{(1)}(\rho N a^{-1}\tau) f(\tau)\,\mathrm{d}\tau + f(R)\frac{R^2 a}{\rho N} h_{n+1}^{(1)}(\rho N a^{-1} R)\right|$$
$$= \mathrm{O}[\epsilon\rho^{-2}\exp(\mathrm{i}\rho N a^{-1}R)], \qquad \text{if } 0 \leqslant \arg\rho \leqslant \pi, \tag{3.23}$$

and

$$\left|\int_R^a \tau^2 h_n^{(2)}(\rho N a^{-1}\tau) f(\tau)\,\mathrm{d}\tau + f(R)\frac{R^2 a}{\rho N} h_{n+1}^{(2)}(\rho N a^{-1} R)\right|$$
$$= \mathrm{O}[\epsilon\rho^{-2}\exp(\mathrm{i}\rho N a^{-1}R)] \qquad \text{if } \pi \leqslant \arg\rho \leqslant 2\pi. \tag{3.24}$$

Let $\bar{\delta}$ be an arbitrarily small fixed positive number. We then obtain for the O terms at the right-hand sides of equations (3.10a) to (3.10d)

$$\mathrm{O}\Big(\rho^2 j_n(\rho N a^{-1} R)\int_0^R \tau^2 j_n(\rho N a^{-1}\tau) f(\tau)\,\mathrm{d}\tau \,.\, \exp(2\mathrm{i}\rho N)\Big)$$
$$= \begin{cases} \mathrm{o}(\rho^{-1}), & \bar{\delta} \leqslant \arg\rho \leqslant \pi - \bar{\delta}, \\ \mathrm{O}(\rho^{-1}), & 0 \leqslant \arg 0 \leqslant \bar{\delta}, \qquad \pi - \bar{\delta} \leqslant \arg\rho \leqslant \pi, \end{cases} \tag{3.25a}$$

$$\mathrm{O}\Big(\rho^2 j_n(\rho N a^{-1} R)\int_0^R \tau^2 j_n(\rho N a^{-1}\tau) f(\tau)\,\mathrm{d}\tau \,.\, \exp(-2\mathrm{i}\rho N)\Big)$$
$$= \begin{cases} \mathrm{o}(\rho^{-1}), & \pi + \bar{\delta} \leqslant \arg\rho \leqslant 2\pi - \bar{\delta}, \\ \mathrm{O}(\rho^{-1}), & 2\pi - \bar{\delta} \leqslant \arg\rho \leqslant 2\pi, \qquad \pi \leqslant \arg\rho \leqslant \pi + \bar{\delta} \end{cases} \tag{3.25b}$$

$$\mathrm{O}\Big(\rho^2 j_n(\rho N a^{-1} R)\int_R^a \tau^2 j_n(\rho N a^{-1}\tau) f(\tau)\,\mathrm{d}\tau \,.\, \exp(2\mathrm{i}\rho N)\Big)$$
$$= \begin{cases} \mathrm{o}(\rho^{-1}), & \bar{\delta} \leqslant \arg\rho \leqslant \pi - \bar{\delta}, \\ \mathrm{O}(\rho^{-1}), & 0 \leqslant \arg\rho \leqslant \bar{\delta}, \qquad \pi - \bar{\delta} \leqslant \arg\rho \leqslant \pi. \end{cases} \tag{3.25c}$$

$$\mathrm{O}\Big(\rho^2 j_n(\rho N a^{-1} R)\int_R^a \tau^2 j_n(\rho N a^{-1}\tau) f(\tau)\,\mathrm{d}\tau \,.\, \exp(-2\mathrm{i}\rho N)\Big)$$
$$= \begin{cases} \mathrm{o}(\rho^{-1}), & \pi + \bar{\delta} \leqslant \arg\rho \leqslant 2\pi - \delta, \\ \mathrm{O}(\rho^{-1}), & 2\pi - \bar{\delta} \leqslant \arg\rho \leqslant 2\pi, \qquad \pi \leqslant \arg\rho \leqslant \pi + \bar{\delta} \end{cases} \tag{3.25d}$$

Hence, combination of equations (3.2a) to (3.2c), (3.10a) to (3.10d) and (3.22) to (3.25d) yields:

$$\left|\frac{f_j(\rho, R)}{\psi(\rho, n)} - f_{j+1}(\rho, R) + f(R)\frac{a^3}{2\rho N^3}\right|$$
$$= \begin{cases} \mathrm{O}(\epsilon\rho^{-1}), & \text{if } \bar{\delta} \leqslant \arg\rho \leqslant \pi - \bar{\delta},\ \pi + \bar{\delta} \leqslant \arg\rho \leqslant 2\pi - \bar{\delta}, \\ \mathrm{O}(\rho^{-1}), & \text{if } -\bar{\delta} \leqslant \arg\rho \leqslant +\bar{\delta},\ \pi - \bar{\delta} \leqslant \arg\rho \leqslant \pi + \bar{\delta}. \end{cases} \tag{3.26}$$

Combination of (3.6), (3.7), and (3.26) leads to:

$$\left|I_j(R,c)-\frac{1}{2\pi \mathrm{i}}\oint_{|\rho|=c} f(R)\frac{a^3}{2\rho N^3}\,\mathrm{d}\rho\right| = \mathrm{O}(\bar{\delta})+\mathrm{O}(\epsilon), \qquad j=1,2. \quad (3.27)$$

Therefore, because the numbers $\bar{\delta}$ and ϵ can be chosen arbitrarily small, letting c tend to infinity we obtain from equation (3.27):

$$\lim_{c\to\infty} I_1(R,c)+I_2(R,c)=\frac{a^3}{N^3}f(R), \quad (3.28)$$

and combination of (3.8) and (3.28) leads to the desired expansion formula:

$$f(R)=\frac{2N^3}{2a^3}\sum_s \frac{\phi(\rho_{sn})\rho_{sn}^2}{(\psi(\rho_{sn},n))'} j_n(k_{sn}R)\int_0^a \tau^2 j_n(k_{sn}\tau)f(\tau)\,\mathrm{d}\tau. \quad (3.29)$$

4. Completeness of the set of modes (2.4) if the conductivities are non-zero

The completeness of the modes (2.4) has been shown hitherto with the assumption that both the sphere and the surrounding medium are non-conductive. In this section we will prove the completeness of the modes (2.4) if the conductivities are not zero.

4.1. Calculational procedure

From the relations

$$k_j^2=\epsilon_j\mu_j\omega^2+\mathrm{i}\sigma_j\omega, \qquad j=1,2, \quad (4.1)$$

we derive:

$$k_2^2(k_1)=\epsilon_2\mu_2\left(\frac{-\mathrm{i}\sigma_1\pm(-\sigma_1^2+4\epsilon_1\mu_1k_1^2)^{1/2}}{2\epsilon_1\mu_1}\right)^2+\mathrm{i}\sigma_2\left(\frac{-\mathrm{i}\sigma_1\pm(-\sigma_1^2+4\epsilon_1\mu_1k_1^2)^{1/2}}{2\epsilon_1\mu_1}\right). \quad (4.2)$$

The choice between the plus and minus sign occurring in equation (4.2) is totally immaterial for the following calculation, and therefore, just to make the function $k_2(k_1)$ definite, we choose the positive sign.

Let the points b_j, $j=1,2,\ldots$, denote the branch points of the function $k_2(k_1)$ considered as a function of the complex variable k_1. We will now consider contour integrals similar to the one introduced in § 3 but replace the contour $|k_1|=c$ by the contour drawn in figure 1. For the appropriate cut in the complex k_1 plane in order to make the function $k_2(k_1)$ single valued we take a curve connecting the points $\mathrm{b}_1, \mathrm{b}_2, \ldots$, which is drawn in figure 1 as a broken line. We will evaluate the integral

$$I(R,c)=\frac{1}{2\pi\mathrm{i}}\int_{c_1}\left(\frac{f_1(k_1,R)+f_2(k_1,R)}{\psi(k_1,n)}-f_3(k_1,R)-f_4(k_1,R)\right)\mathrm{d}k_1, \quad (4.3)$$

where c_1 is drawn in figure 1. Using the techniques of the previous section we obtain:

$$\left|I(R,c_1)-f(R)-\int_L \frac{f_1(k_1,R)+f_2(k_1,R)}{\psi(k_1,n)}\,\mathrm{d}k_1\right|=\mathrm{o}(1), \qquad \text{if } c\to\infty, \quad (4.4)$$

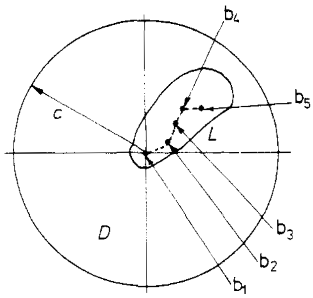

Figure 1. The contour c_1 consists of a large circle with radius c and centre at the origin and a closed curve L. The domain bounded by L contains the branch points b_l but no zero of $\psi(k_1, n)$.

and

$$I(R, c_1)=\frac{N^3}{a^3}\sum_s \frac{k_{sn}^2\phi(k_{sn}, n)}{\psi'(k_{sn}, n)} j_n(k_{sn}R)\int_0^R \tau^2 j_n(k_{sn}\tau)f(\tau)\,\mathrm{d}\tau, \qquad (4.5)$$

where the summation has to be taken over all the singularities of $\psi(k_1, n)$ lying inside the domain D.

We will try to express the integral over L occurring on the left-hand side of equation (4.4) into a series of modes (2.4). The function

$$h(k_1, n)=k_1^2\phi(k_1, n)\exp(-\mathrm{i}k_2(k_1)a)\int_0^a \tau^2 j_n(k_1\tau)f(\tau)\,\mathrm{d}\tau \qquad (4.6)$$

is analytic within the domain E, drawn in figure 2.

Therefore, for every number $\epsilon>0$ there exists a rational function $Q^{(\epsilon)}(k_1)$ with poles lying outside E, such that for all numbers $k_1\in E$ (Saks and Zygmund 1952)

$$|h(k_1, n)-Q^{(\epsilon)}(k_1)|<\epsilon. \qquad (4.7)$$

If the functions $H_l^{(\epsilon)}(k_1)$, $l=1, 2, \ldots,$ denote the principal parts of the functions $Q^{(\epsilon)}(k_1)$ around the poles $k=k_l$, then

$$g(k_1)\equiv Q^{(\epsilon)}(k_1)-\sum_l H_l^{(\epsilon)}(k_1) \qquad (4.8)$$

is an analytic function in the complex k_1 plane.

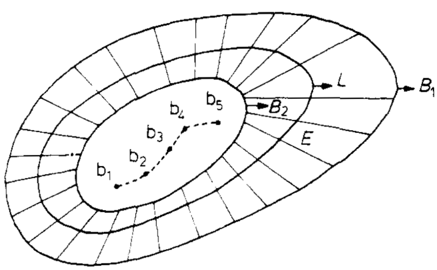

Figure 2. The domain E is bounded by the curves B_1 and B_2 and does not contain a zero of $\psi(k_1, n)$.

Therefore, recalling that no singularities of $(\psi(k_1, n))^{-1}$ are situated within the contour L, combination of equations (4.6), (4.7) and (4.8) yields:

$$\left|\int_L \left[(\psi(k_1, n))^{-1} \exp(\mathrm{i}k_2(k_1)a) j_n(k_1 R)\Big(\sum_l H^{(\epsilon)}(k_1) + g(k_1) - k_1^2 \phi(k_1, n) \exp(-\mathrm{i}k_2(k_1)a) \int_0^a \tau^2 j_n(k_1\tau) f(\tau)\, \mathrm{d}\tau\Big)\right] \mathrm{d}k_1\right| = \mathrm{O}(\epsilon). \tag{4.9}$$

Let M be an arbitrary positive number and suppose that the infinite set of positive numbers λ_j, $j = 1, 2, \ldots$, is bounded by M:

$$\lambda_j < M, \qquad j = 1, 2, \ldots. \tag{4.10}$$

The set of functions $\{\cos(\lambda_j \sqrt{k})\}$ can be shown to be complete with respect to the class of functions which can be approximated arbitrarily closely by a polynomial in powers of k in a bounded domain D (see Lewin 1962, Hoenders and Ferwerda 1974) taking $\mathrm{O}(\nu) \equiv \cos(\lambda_j \sqrt{k})$. The expansion of such a function into this set converges uniformly.

Let the contour L^1 enclose the contour L in such a way that the singular points of the function $H_l^{(\epsilon)}(k)$, which are situated outside L, are also situated outside L^1. The function $H_l^{(\epsilon)}(k)$ is analytic within the simply connected domain bounded by L^1 and, by Runge's theorem, can therefore be approximated arbitrarily closely and uniformly for all values of k situated on L and in the domain bounded by L by a polynomial in powers of k (Saks and Zygmund 1972). Because the set of functions $\{\cos(\lambda_j \sqrt{k})\}$ is complete with respect to the class of functions which can be approximated arbitrarily closely by a polynomial in powers of k in the domain bounded by L to every number $\epsilon > 0$ there exist a positive number $N(\epsilon)$ and numbers $a_j(N)$ such that:

$$\left|\sum_{l_1} H_{l_1}^{(\epsilon)}(k) + g(k) - \sum_{j=1}^{N} a_j(N) \cos(\lambda_j \sqrt{k})\right| = \mathrm{O}(\epsilon). \tag{4.11}$$

The summation over l_1 denotes the (finite) summation over all the singular points outside L. Equations (3.2a), (3.3) and (4.10) lead to:

$$\left|\frac{\exp(ik_2(k_1)) \cos(\lambda_j \sqrt{k_1})}{\psi(k_1, n)} j_n(k_1 R)\right| = \mathrm{O}\{k_1^{-1} \exp[-\tfrac{1}{2}(a - R)|k_1 \sin(\arg k_1)|]\}, \qquad \text{if } |k_1| \to \infty. \tag{4.12}$$

Because $H_{l_2}^{(\epsilon)}(k_1) = \mathrm{O}(k_1^{-1})$ if l_2 labels a singularity k_{l_2} inside L, we derive:

$$\left|\frac{\exp(\mathrm{i}k_2(k_1))}{\psi(k_1, n)} H_{l_2}^{(\epsilon)}(k_1) j_n(k_1 R)\right| = \mathrm{O}\{k_1^{-1} \exp[-(a - R)|k_1 \sin(\arg k_1)|]\}. \tag{4.13}$$

If the contour L is transformed into a circle with infinite radius equations (4.9), (4.11), (4.12) and (4.13) lead to:

$$\left|\int_L \frac{f_1(k_1, R) + f_2(k_1, R)}{\psi(k_1, n)} \mathrm{d}k_1 - \sum_s \exp(\mathrm{i}k_2(k_{sn})) j_n(k_{sn} R)(\psi'(k_{sn}, n))^{-1} \times\Big(\sum_{l_2} H_{l_2}^{(\epsilon)}(k_{sn}) + \sum_j a_j(N) \cos(\lambda_j \sqrt{k_{sn}})\Big)\right| = \mathrm{O}(\epsilon), \qquad \text{if} R < a. \tag{4.14}$$

Equations (4.4), (4.5) and (4.14) show that every function $f(R)$ which is of bounded variation can be approximated arbitrarily closely by a suitable linear combination of the set of functions (2.4) if $0 \leq R < a$. This proves the completeness of the set of functions (2.4).

5. Discussion

The problem which is considered in this paper essentially concerns the extension of classical Sturm–Liouville theory (Courant and Hilbert 1966) to the case where the boundary condition explicitly contains the eigenvalue. Though it seems that such problems do not occur very frequently in physics it has been pointed out in the introduction that they are rather rule than exception, i.e. these problems always occur in connection with fields satisfying a continuity condition across the boundary of two media, like the electromagnetic continuity conditions $\boldsymbol{n} \times \boldsymbol{E}$ and $\boldsymbol{n} \times \boldsymbol{H}$ continuous, and the quantum mechanical continuity conditions ψ and $\partial\psi/\partial n$ continuous across a boundary.

After having shown the completeness of the eigenfunctions the question arises whether or not this set is overcomplete. The overcompleteness of this set of modes can be shown rather easily if the sphere and the surrounding media are not conducting (see the appendix).

The same situation shows up in a similar problem, first formulated by Siegert (1939). Siegert considered the set of natural modes satisfying the radial Shrödinger equation and a continuity condition. The continuity condition arises from the requirement of continuity of the logarithmic derivative of the wavefunction across the boundary of the range of the potential, which is supposed to be delimited by a sphere with radius a.

This set of natural modes can also be shown to be overcomplete (Hoenders 1978). I is also possible to show, using the same kind of analysis to that which has bee i developed in the appendix, that the set of natural modes arising in connection with th e problem of Morse and Feshbach (see the introduction), is overcomplete.

In view of all these examples we conjecture that *the natural modes occurring in electromagnetic theory, as defined by Wolf and Pattanayak* (1976) *and Agarwal* (1973), *or arising in other fields of physics, are overcomplete.*

We shall give a discussion concerning the physical consequences of the overcompleteness of the set of natural modes analysed in this paper and conclude that it is highly probable that for physical fields overcompleteness does not exist at all.

Overcompleteness means that we can dispose of a certain number of modes, or to speak in a colloquial way, 'nature wastes modes'. However, the field vectors of the electromagnetic field defined by equations (2.1a) to (2.1f) are quantities with non-zero imaginary parts and cannot therefore represent physically existing fields.

To elucidate the situation we will consider the modes of the radial magnetic field vector H_r. By virtue of the relation (4.1) we consider equation (2.2b) as an equation in the variable ω. Changing ω into $-\omega$ and taking the complex conjugate we infer that if ω_{sn} is a root of equation (2.4) $-\omega_{sn}^*$ is a root too. Recalling the dispersion relation $\epsilon\mu\omega^2 + i\sigma\omega = k^2$, we observe that if ω_{sn} changes into $-\omega_{sn}^*$ the wavenumber k_{sn} changes into $-k_{sn}^*$ or $+k_{sn}^*$. If however $\exp(ik_{sn}r - i\omega_{sn}t)$ represents an outgoing wave, we have to take $-k_{sn}^*$ because in that case $\exp(i\omega_{sn}^*t - ik_{sn}^*r)$ represents an outgoing wave too. We will show that for a physically existing field a linear superposition of two modes of the radial magnetic field vector H_r, corresponding to the wavenumbers k_{ln} and k_{ln}^*, is indistinguishable from the contribution of just one mode.

The proof of this statement follows immediately from equation (2.1*d*) and

$$\operatorname{Re}[A_1 j_l(k_{ln}r)\exp(-\mathrm{i}\omega_{ln}t)+A_2 j_l(-k^*_{ln}r)\exp(\mathrm{i}\omega^*_{ln}t)]$$
$$=\operatorname{Re}[A_1+(-1)^l A_2^*)j_l(k_{ln}r)\exp(-\mathrm{i}\omega_{ln}t)],$$

if A_1 and A_2 are two arbitrary complex numbers.

We can therefore dispose of one half of the set of modes, for instance those modes for which $\operatorname{Re}(k_{sn})<0$. Relaying on the results obtained by Paley and Wiener (1934) for similar problems, in my opinion it seems highly probable that the set of modes with $\operatorname{Re}(k_{sn})>0$ is just complete.

However, though this completeness problem is certainly interesting from a mathematical point of view there is no reason from the point of view of the physicist to investigate the completeness of the set of modes (2.4) with $\operatorname{Re}(k_{sn})\gtrless 0$. The full set of modes (2.4) is overcomplete and the physical problem does not lead to a condition like $\operatorname{Re}(k_{sn})$ smaller or larger than zero. It is therefore possible to expand the field into a series of natural modes, satisfying the continuity conditions of the field. Because this is possible, there is not much need for the physicist to investigate the completeness properties of subsets of the set (2.4).

Acknowledgments

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Appendix

By virtue of the relation (4.1) we consider equation (2.2*b*) as an equation in the variable ω. Changing ω into $-\omega$ and taking the complex conjugate we infer that if ω_{sn} is a root of equation (2.2*b*) then $-\omega^*_{sn}$ is a root too. Recalling the dispersion relation (4.1) we observe that if ω_{sn} changes to $-\omega^*_{sn}$ the wavenumber k_{sn} changes to $-k^*_{sn}$ or $+k^*_{sn}$. If however $\exp(ik_{sn}r-\mathrm{i}\omega_{sn}t)$ represents an outgoing wave, we have to take $-k^*_{sn}$ because in that case $\exp(\mathrm{i}\omega^*_{sn}t-ik^*_{sn}r)$ represents an outgoing wave too.

The finite set of wavenumbers k_{ln} with $\operatorname{Re}(k_{ln})>0$ and $l=1,\ldots,L$ is arbitrarily chosen from the infinite set of wavenumbers $\{k_{sn}\}$.

Let

$$I(b,n,R)=\frac{1}{2\pi\mathrm{i}}\oint_{|k|=c_n}\prod_{l=1}^{L}\left(\frac{k-k_{ln}}{k-k^*_{ln}}\right)\frac{\exp(\mathrm{i}\rho)j_n(kR)}{\psi(k,n)}F_m(b,k)\,\mathrm{d}k, \qquad \text{if } R<a \tag{A.1}$$

where

$$F_m(b,k)=\left(\frac{\partial}{\partial b}\right)^m\left[\frac{\psi(b,n)}{k-b}\prod_{l=1}^{L}\left(\frac{b-k^*_{ln}}{b-k_{ln}}\right)\right], \qquad b\neq k_{sn}. \tag{A.2}$$

The contours $|k|=c_n$, $n=1,2,3,\ldots$, tend to infinity if $n\to\infty$ and pass between two consecutive poles of the integrand of equation (A.1). Equations (2.3) and (3.3) show that:

$$\lim_{n\to\infty} I(b,n,R)=0, \qquad R<a. \tag{A.3}$$

The residue at the point $k = b$ is immediately calculated and is equal to $\partial^m j_n(bR)/\partial b^m$ if we are permitted to change differentiation and integration. It is for this reason that we introduced the function $F_m(b, k)$, which was introduced into complex analysis by Filon (1906) for the case $b = 0$. (For an explanation of Filon's theory see Watson (1966) and Hoenders and Ferwerda (1974).)

The exact calculation of the residue at $k = b$ is performed with the Laurent expansion of the function

$$F^{(1)}(b, k) = \prod_{l=1}^{L} \left(\frac{k - k_{ln}}{k - k_{ln}^*}\right) \frac{F_m(b, k)}{\psi(k, n)} \tag{A.4}$$

around the point $k = b$:

$$F^{(1)}(b, k) = \frac{m!}{(k - b)^{m+1}!} + \sum_n c_n (k - b)^n. \tag{A.5}$$

Because the principal part of (A.5) only contains the power $(k - b)^{-m-1}$ we obtain

$$\operatorname{Res}(I(b, n, R) : k = b) = \frac{\partial^m}{\partial b^m} j_n(bR). \tag{A.6}$$

The theorem of residues and equation (A.6) yield:

$$\sum_{\substack{s \\ s \neq l}} \prod_{l=1}^{L} \left(\frac{k_{sn} - k_{ln}}{k_{sn} - k_{ln}^*}\right) \frac{j_n(k_{sn}R)}{\psi'(k_{sn}, n)} F_m(k_{sn}, b)$$
$$+ \sum_{l=1}^{L} \frac{j_n(k_{ln}^* R)}{\psi(k_{ln}^*, n)} F_m(k_{ln}^*, b) \prod_{l' \neq l} \left(\frac{k_{ln}^* - k_{l'n}}{k_{ln}^* - k_{l'n}^*}\right)(k_{ln}^* - k_{ln})$$
$$= \frac{\partial^m}{\partial b^m} j_n(bR). \tag{A.7}$$

Recalling that if $j_n(k_{ln}R)$ is a natural mode then $j_n(-k_{ln}^*R)$ is a natural mode too, and using the relation $j_n(k_{ln}^*R) = (-1)^n j_n(-k_{ln}^*R)$, equation (A.7) shows that each function of the set

$$\left\{\frac{\partial^m}{\partial b^m} j_n(bR);\ m = 0, 1, 2, \ldots\right\}, \qquad b \neq k_{sn},\ 0 \leqslant R < a \tag{A.8}$$

can be expanded into the *reduced* set of natural modes

$$\{j_n(k_{sn}R);\ s \neq l,\ l = 1, 2, \ldots, L\}. \tag{A.9}$$

Therefore if the set of functions (A.8) can be shown to be complete in the interval $0 \leqslant R \leqslant a$ we have proved the overcompleteness of the set of functions (2.4). However, this set of functions is the set of coefficients of the Taylor series expansion at the point $\lambda = b$ of the function $j_n(\lambda R)$:

$$j_n(\lambda R) = \sum_{m=0}^{\infty} \frac{1}{m!} (\lambda - b)^m \frac{\partial^m}{\partial b^m} j_n(bR). \tag{A.10}$$

The series (A.10) converges uniformly for all values of R with $0 \leqslant R \leqslant a$ and shows that each function $j_n(\lambda_{ln}R)$ of the complete set $\{j_n(\lambda_{ln}R)\}$, where the numbers λ_{ln} are the roots of the transcendental equation $j_n(\lambda a) = 0$, can be approximated arbitrarily closely and uniformly in the interval $0 \leqslant R \leqslant a$ by a suitable linear combination of functions (A.8). This result and (A.7) proves the overcompleteness of the natural modes.

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